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Understanding Point and Slope in Linear Equations and Approximations: A Case Study

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Abstract: This article delves into an intervention designed to enhance a precalculus student's understanding in constructing linear equations and in approximating function values. Before the intervention, the student primarily relied on the algebraic formula $\frac{y_2-y_1}{x_2-x_1}$ to determine slope, knew slope geometrically as $\frac{\Delta y}{\Delta x}$ solely for integer values, and struggled to construct a linear equation without an explicitly shown y-intercept. Through the intervention, the student comprehended slope m as a unit rate and expanded this understanding to $\Delta y = m\Delta x$ for integer Δx by iterating m, Δx number of times. She also successfully used this newfound understanding to construct linear equations, including tangent line equations, and approximate function values. Nonetheless, she faced some challenges in comprehending $\Delta y = m\Delta x$ for non-integer Δx and in conceiving point as a multiplicative object with its coordinates representing covarying quantities. Further exploration is needed to validate these findings and address potential difficulties that students might face in grasping slope in the context of linear equations and approximations.

Keywords: Point, slope, multiplicative object, tangent line, linear approximation

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Introduction

This study discusses the crucial concepts of point and slope, which are essential for comprehending mathematical concepts from algebra to calculus (Carlson et al., 2002; David et al., 2019; Nagle et al., 2019; Thompson & Carlson, 2017). Specifically, it focuses on point and slope in the context of linear equations and graphs in algebra, as well as tangent lines and linear approximations in calculus. Research has found that students often struggle with linear equations and graphs due to their inadequate understanding of point and slope, including rate of change (Birgin, 2012; Knuth, 2000; Moschkovich et al., 1993; Nagle et al., 2019; Thompson et al., 2017). Research has also shown that students have difficulty constructing tangent line equations and solving approximation problems (Amit & Vinner, 1990; Çekmez & Baki, 2016; Moore-Russo & Nagle in this volume; Orton, 1983). However, little research has been conducted on the potential influence of understanding of slope and point on comprehending tangent line equations and linear approximations.

This study was motivated by my observation that calculus students struggled with tangent line equations and approximations of function values, even though they could successfully compute derivatives. I suspected that the difficulties partially stemmed from the students' inability to construct linear equations without explicit *y*-intercepts. Therefore, to investigate students' ability to construct linear equations, I conducted a survey with two precalculus classes. The survey included two conversion problems from a graph to an equation: the first graph displayed the *y*-intercept and another point with integer coordinates in the first quadrant, while the second graph showed two points with integer coordinates in the first quadrant.

The result of the survey confirmed my hypothesis. Out of 52 students, only 6 students provided a correct equation for the second graph, while 31 did for the first graph. Therefore, I designed a teaching experiment (Steffe & Thompson, 2000) and implemented it with three precalculus students, aiming to enhance their understanding of slope and line. In this article, I report on the results of the implementation, addressing the research question: How do precalculus students understand slope and point and use this understanding to construct linear equations and approximate function values? I will begin by discussing the theoretical and empirical foundations of this study.

Theoretical and Empirical Background

Research has shown that many individuals have an underdeveloped schema of Cartesian space, consisting at best of only the axes and points (Herscovics, 1989; Trigueros & Martinez-Planell, 2010). They may be able to plot points in a plane by making horizontal or vertical moves or read off coordinates from given points in simple tasks (Kerslake, 1981; Padilla et al., 1986). However, they often have a hard time distinguishing values from points, leading them to represent a value such as $f(2)$ as a point $(2, f(2))$ (Thompson et al., 2017). Moreover, they struggle to plot points and sketch graphs when describing dynamic movements of objects as graphs (Frank, 2016; Stalvey & Vidakovic, 2015; Thompson et al., 2017).

Point as Multiplicative Object

Individuals' struggles, as described above, often result from their inability to conceive point as a multiplicative object (Saldanha & Thompson, 1998). According to Thompson et al. (2017),

> [w]hen someone conceives a generic point on a graph, either in retrospect or anticipation, so that its coordinates represent a state of two quantities' covariation, she has conceived the point as a multiplicative object and the graph containing it as a record of the quantities' covariation. … On the other hand, when someone understands the coordinates of a generic point on a graph as "over this much and up that much", he is conceiving the point's coordinates as a recipe for locating the point (p. 1307).

Due to the emphasis on quantities and covariation in the notion of point as a multiplicative object, studies that use this notion are mostly conducted with contextualized tasks. Some studies have investigated individuals' understanding of graphs of dynamical objects moving in a plane (Frank, 2016; Thompson et al., 2017), or of parametric functions related to the volume and height of bottles (Stalvey & Vidakovic, 2015). Other studies have analyzed individuals' comprehension of exponential growth (Ellis et al., 2015) with quantities of height and time, or of rate of change with covarying quantities of area of square and temperature (Johnson, 2012; Stalvey & Vidakovic, 2015). Yet other studies have used this notion as a theoretical basis for task design (Stevens et al., 2017) or the development of another framework of static and emergent shape thinking (Moore & Thompson, 2015).

Unlike the other studies mentioned, this study employs the notion in non-contextualized tasks with the aid of the history of numbers. Even if a function equation, such as $y = f(x)$ with x and y denoting variables, has no real-life quantities attached with it, it is still possible to reason with quantities. Indeed, variables indicate varying numbers, and numbers indicate quantities of magnitudes (Lakoff & Núñez, 2000; Smith & Confrey, 1994; Thompson et al., 2017).

Line, Slope, and APOS Theory

Students often struggle with linear equations and graphs as they fail to discriminate between *x* and *y* intercepts; switch *x* and *y* coordinates when plotting, omit *x* in the linear equation, or do not understand the role of *b* in $y = mx + b$ in graphs (Barr, 1980; Birgin, 2012; Cho & Nagle, 2017; Moschkovich,1996). Additionally, students may compute slope using "run over rise" instead of "rise over run," disregard the direction of rise and run, mix up the coordinates when using the formula $\frac{y_2-y_1}{x_2-x_1}$, represent a linear equation with a negative slope as an increasing line, or fail to determine slope when a linear equation is not in the slope intercept form, $y = mx + b$ (Birgin, 2012; Cho & Nagle, 2017; Kondratieva & Radu, 2009). While some of these errors may stem from procedural issues or other factors, many of them can be attributed to students' insufficient, conceptual understanding of slope (Birgin, 2012; Cho & Nagle, 2017; Kondratieva & Radu, 2009). Indeed, slope is a complex concept that requires the connections among multiple representations and has multiple facets that are expressed differently in various situations (Cho & Nagle, 2017; Johnson, 2015; Lobato, Ellis, & Muñoz, 2003; Nagle et al., 2013; Nagle et al., 2019; Moore-Russo et al., 2011; Stump, 2001).

Meanwhile, Nagle et al. (2019) explained eleven conceptualizations of slope using the APOS framework (Arnon et al., 2014). In this study, I utilize three of these conceptualizations that are relevant to my study: algebraic ratio (AR), geometric ratio (GR), and the function property (FP). Below is a summary of Nagle et al.'s description of the action and process stages, as well as the transition level from action to process for the three conceptualizations.

Someone with an action conception of slope as an AR can compute slope using the formula $\frac{y_2-y_1}{x_2-x_1}$, but she has no geometric referent to justify $\frac{y_2-y_1}{x_2-x_1} = m$. At a transition level of slope as AR, she may know that slope is independent of two specific points and can apply this understanding to a fixed point (x_1, y_1) and any point (x, y) on the line. However, she may only understand it algebraically and be unable to geometrically justify the equation $\frac{y-y_1}{x-x_1} = m$. Additionally, at a transition level, she may have "interiorized properties of real numbers into symbolic manipulation processes" (Nagle et al., 2019, p. 5) as AR and imagine obtaining $y-y_1 = m(x-x_1)$ from the formula, $\frac{y-y_1}{x-x_1} = m$.

In contrast, someone who holds an action conception of slope as GR perceives slope as a static image of rise (Δy) over run (Δx) and typically determines slope by counting grids. Consequently, she may not keep track of the directions in rise and run nor comprehend that Δy varies as Δx varies at the same rate. At a transition level of slope as GR, she may visualize slope with dynamic images via different sized similar triangles. She may recognize that slope is independent of the triangle's size and apply this understanding to describe the behaviors of lines, such as increasing or decreasing. Yet she may not have an algebraic referent to justify the geometric ideas.

When someone connects $\frac{y-y_1}{x-x_1}$ and $\frac{\Delta y}{\Delta x}$ with understanding that Δy has the same meaning as $y - y_1$ and Δx as $x - x_1$, and has a transition level of slope as both AR and GR, she is moving towards a process stage of slope. To attain a process stage, however, she needs to comprehend slope as a rate of change—the change in output is *m* when the change in input is 1, a transition level of FP—by linking FP with both AR and GR. Additionally, with the unit rate as GR and a dynamic imagery of slope, she needs to understand $\Delta y = m\Delta x$ geometrically for any Δx. To achieve a process stage, she also needs to be able to explain the algebraic equation $y-y_1 = m(x-x_1)$ using $\Delta y = m\Delta x$, without explicitly computing or drawing.

Tangent Lines and Approximation of Function Values

The concept of tangent lines is crucial in mathematics, as it is related to other fundamental concepts such as derivative and linear approximations. However, students often develop misconceptions and inconsistent concept images (Vinner, 1983) of tangent lines, due to the multitude of terms and definitions associated with the concept (Sierpinska, 1985; Vinner, 1991) as well as the complexity of the derivative concept involved (Asiala et al., 1997; Orton, 1983; Zandieh, 2000). Most of the research on tangent lines, such as the studies mentioned above, has primarily focused on how students determine the existence of tangent lines at specific points (such as cusps or points of inflection) in various curves, or how they define or conceive of the tangency in multiple contexts, such as geometry and calculus. However, there has been little research conducted on the challenges students face when constructing tangent line equations.

Studies have shown that students typically use the point-slope form of a linear equation, $y - f(x_1) = f'(x_1)(x - x_1)$, or its treated form, $y = f'(x_1)(x - x_1) + f(x_1)$, to construct tangent line equations (Amit & Vinner, 1990; Biza & Zachariades, 2010; Çekmez & Baki, 2016; Orton, 1983). However, constructing tangent line equations is not easy for students, with only a

31% success rate reported among a group of high school students (Biza & Zachariades, 2010). Students' difficulties with tangent lines may be due to a lack of skills in computing derivatives (Orton, 1983) or an insufficient understanding of the relationship between the derivative and the slope of the tangent line (Amit & Vinner, 1990). However, it is possible that their ability to construct tangent lines is also affected by a lack of familiarity or ability to work with the point-slope form, as many individuals only have the slope-intercept form in their concept image (Vinner, 1983) of slope (Nagle & Moore-Russo, 2013) and rely heavily on it when constructing linear equations (Cho & Nagle, 2017).

Linear approximation of a function, $y = f(x)$, involves approximating the function with its tangent line, $y = L(x) = f'(x)(x - x_1) + f(x_1)$, at the point of tangency, $(x_1, f(x_1)$. Consequently, when x is close to x_1, $f(x)$ can be estimated by $L(x)$. Previous studies have indicated that students struggle with linear approximations and approximations of function values even if they are able to compute derivatives, (Amit & Vinner, 1990; Moore-Russo & Nagle in this volume). However, research on student understanding of linear approximation is limited and does not offer much insight into the kinds of understanding or difficulties students have with the concept, highlighting a need for further research.

Methodology

Setting

The participants of the intervention were three precalculus students at a small state university located in the southeastern region of the US. I reached out to precalculus instructors to request recommendations for their students and sent an invitation email to all 15 recommended students. Among them, three students responded to my invitation and agreed to participate in the intervention. They were compensated at a rate of $10 per hour for their participation. For this study, I primarily focus on Tanya (pseudonym), one of the three participants. Tanya was a local high school student enrolled in our dual enrollment program. She had completed Algebra 2 at her high school and had not previously taken precalculus or calculus.

All precalculus classes in the university utilized the same textbook, *Algebra and trigonometry* (Abramson, 2018), and followed a uniform syllabus. The topic of linear functions was typically covered in one session, lasting approximately one and a quarter hour, at the beginning of the semester. The textbook primarily uses the formula, $\frac{y_2 - y_1}{x_2 - x_1}$, to explain the slope between two points on a line. It also explains slope geometrically, mainly as move over and up, and states $\frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x}$ with minimal explanation. In addition, it employs the move over and up technic to convert between linear equations and graphs with rational slopes and integer-coordinated y-intercepts. The text also introduces the point-slope form, $y-y_1 = m(x-x_1)$ as an equation of a line passing through the point (x_1, y_1) with a slope of m. However, it quickly transforms the form into $y = mx + b$, with little explanation linking the slope $\frac{y-y_1}{x-x_1} = m$ and the equation $y = mx + b$.

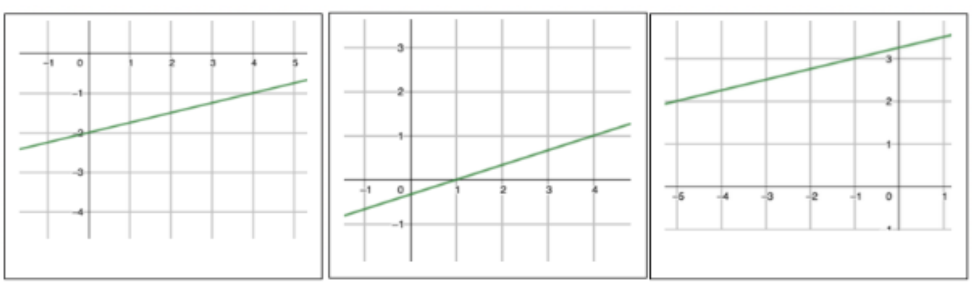
Participants were interviewed for a total of 5-7 sessions at the end of the semester, after having learned linear functions. Each interview session lasted approximately one and a half hours. During the first one to two sessions, I utilized a semi-structured clinical interview form (Ginsburg, 1997) to investigate their understanding of slope as well as their ability to construct linear and quadratic equations and graphs. For the remaining sessions, I used a teaching experiment form (Steffe & Thompson, 2000) to intervene in their understanding of the concepts. As the interviews progressed, I modified tasks based on their responses. The tasks shown in Figures 1, 2, and 3 are examples of the implemented tasks and primarily utilized in this paper for the analysis of Tanya's understanding.

Instrument

Figure 1

Task on Linear Equations

(a) (b) (c)

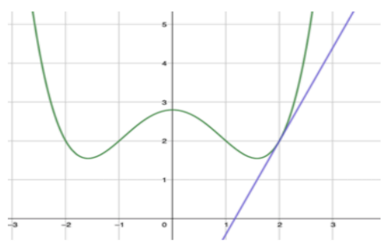


The Figure 1 task aimed to investigate and intervene in their use of slope in constructing linear equations. Based on past studies (Amit & Vinner, 1990; Nagle & Moore-Russo, 2013; Cho & Nagle, 2017) and the results of my survey, I hypothesized that the participants would face challenges with Figure 1bc due to the absence of explicit y-intercepts in the graphs. The Figure 2 and 3 tasks aimed to investigate and intervene in their ability to construct tangent line equations and approximate function values.

These tasks were specifically designed to allow the participants to respond to the questions based on their knowledge of slope, regardless of their knowledge of derivatives. This ensured that the difficulties the participants encountered in constructing tangent line equations or approximating function values were attributed primarily to their limited understanding of slope and linear equations rather than their understanding of derivatives. Moreover, the complexity of the tasks gradually increased, progressing from providing Δx as an integer to a non-integer, from representing the slope as a numerical value to as a formula, and from providing graphs to not providing any.

Figure 2

Task on Tangent Line and Approximation of Function Values



The *y*=g(x) graph (in blue) is the tangent line of y=f(x) (in green) at the point (2, 2) and the slope of the line g is 12/5.

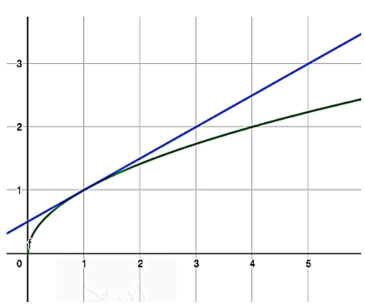
a) Estimate the value of f(2.1).
b) Find the equation of the line: y=g(x).
c) Estimate f(2+h).

Figure 3

More Tasks on Approximation of Function Values

(a)

The line in blue is the tangent line to the y=$\sqrt{x}$ graph in green at *x* = 1. Estimate $\sqrt{1.05}$.



(b)

The slope of the tangent line of $f(x) = 2x^2 + 1$ at $x = a$ is $4a$. Estimate $f(2.1)$.

Estimate $\sqrt[3]{1.05}$ when the slope of the tangent line of $f(x) = \sqrt[3]{x}$ at *x*=*a* is $\frac{1}{3\sqrt[3]{a^2}}$.

Data Analysis

Data analysis commenced at the beginning of the interviews. Following each interview, I viewed the video recordings and created a brief written record of the video content, which included rough transcripts of "interesting" moments. Simultaneously, I developed codes based on the research questions, the study framework, and student understanding observed in the data (Miles & Huberman, 1994). During this preliminary analysis phase, I formulated hypotheses on the student's understandings and difficulties. In the subsequent phase, I constantly compared and revised codes in the spirit of the constant comparative method (Glaser & Strauss, 1967) to confirm, refute, or revise the hypotheses.

Results

The results presented in this section pertain to Tanya's varying levels of comprehension of slope and related ideas. The first segment outlines her understanding and difficulties mainly associated with the Figure 1 task, which is framed as linear equation tasks. The second segment describes her understanding and difficulties mainly associated with the Figure 2 and 3 tasks, which are framed as tangent line and approximation of function value tasks.

Linear Equations and Graphs

Action Stage

Tanya demonstrated an action conception of slope as AR at the beginning of the interviews. For each of the three graphs in Figure 1, she chose two integer-coordinated points on the graph and used the formula, $\frac{y_2-y_1}{x_2-x_1}$, to determine the slope. However, she relied almost entirely on the formula to determine slope, even when it was unnecessary. For instance, when asked to find the slope of $y = 3x + 5$, provided with the two points (4, a) and (6, b) on the line, she found the values of a and b and used the formula instead of identifying the slope from the equation $y = 3x + 5$.

Tanya's concept image (Vinner, 1983) of line solely involved the slope intercept form, $y = mx + b$. When asked to determine an equation representing "a collection of all points (x, y) that satisfy the following condition: the slope between (5, 6) and (x, y) is 3," she wrote $y = 3x + b$ after plotting the points (5, 6) and (6, 9), but was unable to determine the value of b using the two points. Moreover, when asked to interpret the equation, $\frac{y-3}{x-2} = 5$, she confessed that she had seen it before but had never understood it. When prompted again to interpret it, she manipulated $\frac{y-3}{x-2} = 5$ into $y = 5x - 7$ and described it as a line with a slope of *5* and a *y*-intercept of -7.

Due to her action conception of slope as AR and her reliance on the slope-intercept form, Tanya could only construct a linear equation for the graph in Figure 1a where the *y*-intercept was explicitly shown. However, when the *y*-intercepts were not explicitly displayed, as in the graphs of Figures 1b and 1c, she provided the linear equations as $y = \frac{1}{3}x - 0.3$ and $y = \frac{1}{4}x + 3.3$ by visually estimating the value of *b*, both of which were incorrect.

Transitional Level from Action to Process Stage

As she had difficulty with graphs in Figure 1b and 1c, I prompted her to determine the value of *b* in Figure 4 to intervene in her understanding. She constructed the equation, $\frac{1}{4} = \frac{b-(-1)}{3-4}$, as an equivalence between two slopes—1/4 as slope as AR between (0, -2) to (4, -1) and $\frac{b-(-1)}{3-4}$ between (4, -1) and (3, *b*), showing a transition level of slope as AR. I then prompted her with some questions that might activate her understanding of slope as FP. See Excerpt 1 for the conversation between Tanya and me. I was the interviewer (I) and Tanya was the participant (T).

When prompted what the slope of ¼ meant, Tanya initially described it as "[y]ou go up once and go over 4 times"—showing an action stage of slope as GR (lines 1-3, Excerpt 1). She also knew that when run was 1, rise was ¼ (lines 4-7). However, when asked to use this understanding to find the value of *b* in (3, *b*), she was unable to apply the knowledge. Her knowledge of $\frac{1}{4} = \frac{\frac{1}{4}}{1}$ as equivalent ratios was limited to numbers only, and she had not yet made the connection to equivalent ratios of $\frac{\Delta y}{\Delta x}$ (GR) or understood slope as a rate of change (FP) (lines 8-12). With some prompts (lines 14-17), she extended her understanding of $\frac{1}{4} = \frac{\frac{1}{4}}{1}$ as numbers to an understanding of slope as $\Delta x = 1$ and $\Delta y = \frac{1}{4}$ as well as $\Delta x = -1$ and $\Delta y = -\frac{1}{4}$—a transition level of slope as GR and FP. She then used the understanding to determine *b* by drawing the horizontal and vertical segments from (4, -1) to (3, *b*) and basing on the reference point of (4, -1), while saying "it must be 1/4 smaller than -1."

Afterwards, I asked her to construct an equation for Figure 1c, a task she had previously failed. She selected the reference point (-1, 3) and determined the *y*-intercept using the reference point and the unit rate. She said, "from here, I go over 1 and up 1/4, so *b* must be ¼ more than

3," with no work other than adding 3 and ¼ as fractions. Her work seemed to indicate that she conceived a point as a multiplicative object (Saldanha & Thompson, 1998) since she consistently determined a new point—(3, *b*) or the *y*-intercept— based on the reference point with the expectation that "its coordinates represent a state of two quantities' covariation" (Thompson et al., 2017, p. 1307). However, in subsequent tasks, she showed that her understanding of the point as a multiplicative object was uncertain.

Excerpt 1

I: You said the slope of this line is 1/4 (pointing Figure 1a). What is the meaning of the slope of ¼.

T: You go up once and go over 4 times.

I: OK. Then, if run is 1, what is rise?

T: ¼

I: How do you know?

T: Because ¼ is ¼ over 1.

I: Ok. Then, can you use that understanding to determine the coordinates of this point (the point (3, *b*) in Figure 4)

T: The rise is 1 and the run is 4. (She stopped talking and did not seem to know how to move forward.)

I: You said that the slope of ¼ means for every increase of 4 in *x*, there is an increase of in *y*.

T: Yes, every time when *x* is increased by 4, *y* is increased by 1.

I: Yeah. So, every time *x* is increased by 1, *y* is increased by what?

T: ¼

I: OK. What if *x* is decreased by 1? What happens to *y*?

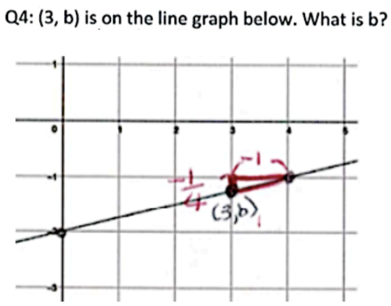
T: Then, *y* is decreased by ¼.

I: Ok. Then can you use the understanding to find the value *b* here (in Figure 4)?

T: Ok. So, *x* is decreased by 1 and *y* is decreased by ¼ (she drew horizontal and vertical segments and wrote -1 and -1/4, respectively). So, it must be -1-1/4.

Figure 4

Slope in Linear Equation and Its Use in Approximation

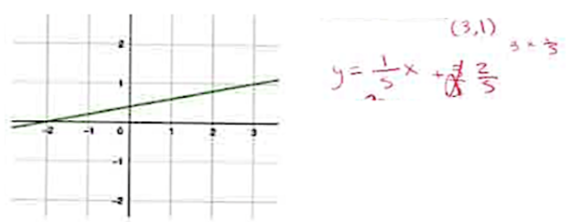


Issue in Conceiving Point as a Multiplicative Object

When determining the linear equation for the graph in Figure 5, she found the slope of the line as GR by simply inspecting the graph. She then wrote $y = \frac{1}{5}x +$ (with the blank replacing b in her mind) and tried to determine b. She said, “each time I move left by 1, I need to go down by 1/5. So, I need to go down by 3 times 1/5,” while writing $3 \cdot \frac{1}{5}$—indicating that she understood Δy as $\frac{1}{5}\Delta x$, with $\frac{1}{5}$ as a unit rate.

Figure 5

Constructing Linear Graph Integrating Point and Slope as Multiplicative Objects



When requested to explain her thought process, she explained how to “locate” the *y*-intercept with the understanding of $\Delta y = m\Delta x$ by saying “I have to know how many times I need to move to get to the *y*-intercept, I mean in *x*. And then I need to take that into the consideration to determine how much move I have to make in *y*.” Yet when she constructed the linear equation, she incorrectly determined the value of *b* using her correct idea. Instead of subtracting $\Delta y = \frac{3}{5}$ from the *y*-coordinate of the reference point (3, 1), she accepted $\Delta y = \frac{3}{5}$ as the value of *b* and claimed $y = \frac{1}{5}x + \frac{3}{5}$ as the equation for the graph. Only after being prompted with questions, such as “where is your original point” and “3/5 down from what,” she realized that her reference point was (3, 1) and replaced 3/5 with the correct value of 2/5 by subtracting $\frac{3}{5}$ from 1 (see Figure 5).

Tanya's confusion between Δy and *b* persisted in a table task where she looked for an equation for the relationship between *x* and *y* (see Excerpt 2 on the task in Figure 6).

Excerpt 2

T: It looks like every time you move over 1 in *x*, move up 6 in *y*.

I: Ok. So, what is the equation?

T: Ok. Here is 3 and 10. Every time you move over once, go up 6. (She wrote *y* = 6*x*.) It is 3 and you have to go back. So go down 18. So . . . (She was silent for a moment and then asked me a question.) What is the formula that we were using that was like *y*-*k* over something?

(Conversation omitted here as it is irrelevant to this article.)

I: I wonder why you stopped there. You said you had to move three times to the left and then 18 down. Why did you stop there and try to use a different method?

T: I stopped because I could not visualize.

I: Ok. Then can you draw the situation?

T: So here is the point (3, 10). I go left by 3 and down by 18.

(She plotted (3, 10) and drew horizontal and vertical segments.)

I: Ok. So, what is the point (pointing the *y*-intercept)?

T:18?

I: You are saying this (horizontal segment) is 3 and this (vertical segment) is 18. Isn't that what you are saying?

T: Yeah, I think so.

I: Ok, then tell me what you are doing in here (pointing the vertical segment).

T: I want to go down 18.

I: From where?

T: Oh, it is from 10. So, it must be 8.

I: Yeah, somehow you stopped in the middle, so I wondered what you were thinking.

T: I didn't know where to take the 18 at (from).

Figure 6

Linear Equation for Tabular Representation

Assuming the same pattern persists throughout the data as shown below, what is y in terms of x?

x	3	4	5	6	7
y	10	16	22	28	34

She found the slope from the table by visualizing the situation as indicated in her words, "you move over 1 in *x*, move up 6 in *y*" and "[i]t is 3 and you have to go back. So go down 18" (lines 1-4, Excerpt 2). She then used her understanding of $\Delta y = m\Delta x$ to locate the *y*-intercept. However, with that understanding, she was unable to determine the coordinates of the *y*-intercept. Instead, she asked for my help in recalling the formula $\frac{y-k}{x-h} = m$ that she had utilized in a previous task with my assistance.

When asked why she needed the formula to construct an equation, she said she could not determine the *y*-intercept because she was unable to visualize the situation (lines 8-10, Excerpt 2). I then suggested her to draw the situation as a graph. She correctly plotted (3, 10) and drew the horizontal and vertical segments from the point (3, 10), stating "go left by 3 and down by 18." However, she incorrectly claimed that *b* in $y = 6x + b$ was 18. Only after being prompted 18 was down "from where," she realized that the reference point was (3, 10) and correctly determined *b* by subtracting 18 from 10 (lines 16-25). It was evident that her struggle did not stem from a deficiency in visualization, as she claimed. Instead, her challenge lay in working with the coordinates of the reference point in the combination of Δx and Δy, as demonstrated by her words, "I didn't know where to take the 18 at."

Tangent Line and Linear Approximation of Function Values

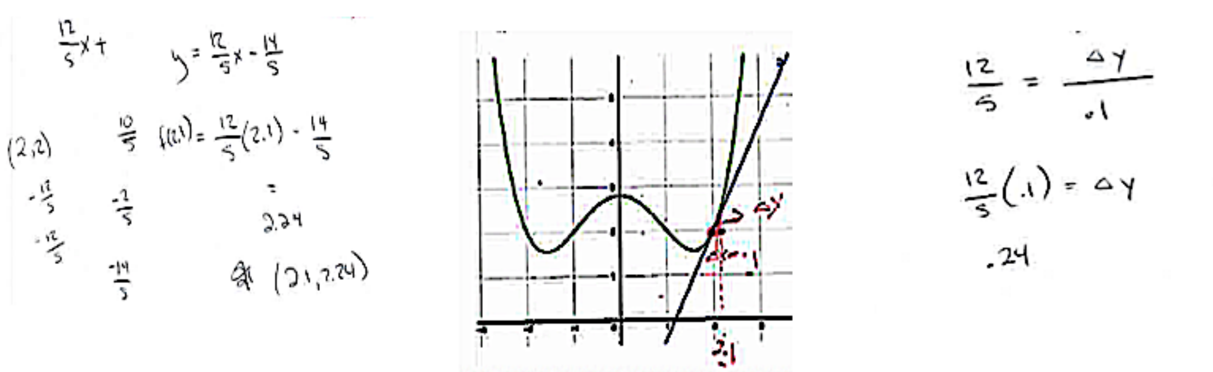
Transitional Level from Action to Process Stage

Before introducing the tangent line and linear approximation tasks, I explained to Tanya that the function curve and its tangent are almost identical around the point of tangency. To illustrate it, I showed her the graphs of $f(x) = x^3$ and $g(x) = 3x - 2$ zoomed in at the point of tangency (1, 1). I also explained to her $f(x)$ could be estimated by $g(x)$, when x is close to 1 due to their closeness around (1, 1). Afterwards, I presented the Figure 2 task and asked her to estimate $f(2.1)$. Tanya correctly wrote an equation, $y = \frac{12}{5}x +$ (with the blank replacing *b*), determined the blank by mentally moving left 1 and down $\frac{12}{5}$ twice from the reference point of (2, 2), and provided a correct tangent line equation, $y = \frac{12}{5}x - \frac{14}{5}$ (Figures 7a and 7b). She then asked, "[y]ou said they are almost equal around here. Right?" When I said yes, she found an estimate of *f*(2.1) by replacing *x* by 2.1 in $\frac{12}{5}x - \frac{14}{5}$. It seemed that her understanding of slope and her conception of point as a multiplicative object were carried over to the linear approximation task without much difficulty.

Figure 7

Estimation of a Function Value

(a) (b) (c)

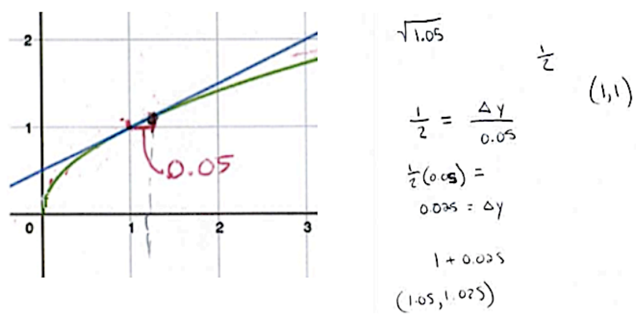


I then requested her to use the geometric approach she had previously used in the linear tasks to approximate *f*(2.1) without relying on the tangent line equation. One advantage of this approach is that it can allow her to extend her understanding of $\Delta y = m\Delta x$ from integer Δx to non-integer Δx. This understanding is connected to $dy = f'(x)dx$ in calculus and is crucial for understanding the concept of derivatives. To guide her in that direction, I added horizontal and vertical segments to the drawing (shown in red in Figure 7b) and asked her to use the drawing to estimate *f*(2.1). She understood my request and said, "I think you want me to find it using slope because slope is supposed to be the same no matter what the point is." She then set up the equation, $\frac{12}{5} = \frac{\Delta y}{.1}$ (showing a transition level as GR) and solved for Δy = .24 from $\Delta y = \frac{12}{5}(.1)$ (see Figure 7c). Afterward, she added .24 to the *y*-coordinate of the reference point (2, 2) and estimated *f*(2.1) to be 2.24—demonstrating her understanding of $\Delta y = m\Delta x$ derived from the computation of $\frac{\Delta y}{\Delta x} = m$. Had she been a calculus student, I would have recommended using *dx* and *dy* in place of Δx and Δy. However, since she was a precalculus student, and our discussion revolved around the tangent line, I chose not to correct her notation during this or any subsequent tasks.

In a subsequent task (Figure 3a), Tanya successfully estimated $\sqrt{1.05}$ using the graphs of $y = \sqrt{x}$ and its tangent at (1, 1) (Figure 8). She voluntarily drew horizontal and vertical segments, set Δx as 0.05, set up the equation ½ = $\frac{\Delta y}{0.05}$, and determined Δy as ½ (.0.05) = 0.025. Afterward, she used the reference point (1, 1) to get an estimate of $\sqrt{1.05}$ as 1 + .025 = 1.025 (see Figure 8). However, as the problems became more complex, she struggled to use the reference points to provide estimates of function values, as shown in the next section.

Figure 8

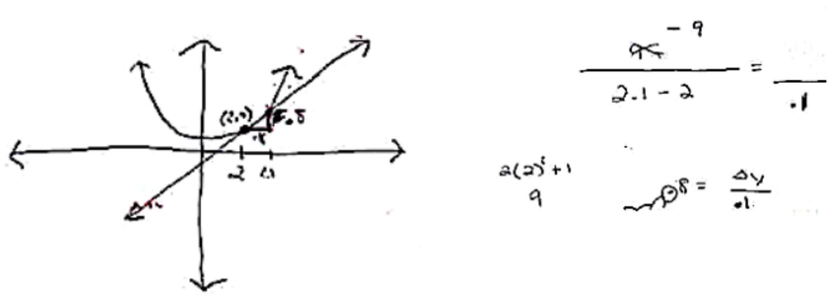
Estimation of a Function Value, $\sqrt{1.05}$



Problem in Conceiving Point as a Multiplicative Object

Figure 9

Estimation of a Function Value, f(2.1) when $f(x) = 2x^2 + 1$ with the Slope of *f* 4*a* at *x* = *a*



In the Figure 2c task, Tanya was asked to estimate $f(2 + h)$. She began by setting up the equation $\frac{12}{5} = \frac{\Delta y}{h}$ and subsequently derived $\Delta y = \frac{12}{5}h$, demonstrating again her need of writing $\frac{12}{5} = \frac{\Delta y}{h}$ to determine Δy. Nevertheless, she struggled to incorporate the changes (Δx and Δy)

into the coordinates of the reference point. As a result, she erroneously claimed $\frac{12}{5}h$ as an estimate of $f(2+h)$. Even after being reminded that she needed an estimation for $f(2+h)$, she continued to incorrectly claim that $2+h+\frac{12}{5}h$ was an estimate of $f(2+h)$. It was only after prompting her about the reference point that she was able to correct her mistake, arriving at a correct estimate of $2+\frac{12}{5}h$.

Excerpt 3

I: Tell me what you just did.

T: Ok. I first had to find this point (referring to (2, 9)). It is 2 times 2 squared plus 1, so it is (2, 9)

I: Ok.

T: And then I know slope is 4.

I: Why is it 4?

T: No, I mean it is 4 times 2.

I: Ok, so the slope is 8.

T: Then, I got the change in *x*, so I am looking for the change in *y*.

I: Ok. So, what is the change in *x*?

T: It is .1. (She added .1 in the drawing.)

I: Ok. So, what is the change in *y*?

T: It is 8, right?

I: The change in *y* is 8?

T: Umm. Yes. (She wrote $.8 = \frac{\Delta y}{.1}$)

I: Hmm. When you say the slope is 8, what do you mean by that?

T: When *x* is changed by 1, *y* is changed by 8.

I: Ok. So, in this case (pointing the triangle in Figure 9), what is the change in *x*?

T: .1

I: Yeah, so what is the change in *y*?

T: Umm. (She removed the decimal point from $.8 = \frac{\Delta y}{.1}$ to change it to $8 = \frac{\Delta y}{.1}$.) It is .8.

Her struggle continued in the first task of Figure 3b, where she determined an estimate of $f(2.1)$ for $f(x) = 2x^2 + 1$, given the slope of the tangent line at $x = a$ as $4a$. She started the task by independently sketching the graphs of $f(x) = 2x^2 + 1$ and its tangent. She then added other components to the graph, marking 2.1 on the *x*-axis, the point of tangency (2, 9), and both horizontal and vertical segments. Subsequently, she set up the equation $\frac{-\ 9}{2.1-2} = \frac{\quad}{.1}$ to represent the slope as AR as depicted in Figure 9. Nevertheless, she became lost during the process and fell silent for some time. To gauge her thought process, I initiated a series of questions.

As indicated in lines 1-8 in Excerpt 3 and Figure 9, she demonstrated understanding of the problem and was able to describe the situation with a graph. She also correctly identified the slope of the line as 8 and Δx as .1. However, she incorrectly claimed that Δy was 8 (see lines 9-15, Excerpt 3). Her miscalculation of Δy was surprising in that she had previously determined Δy successfully, even with non-integer Δx, such as in the approximation of $\sqrt{1.05}$. One possible explanation for her confusion is that her drawing lacked scaling and grid lines, possibly affecting her perception of the magnitude of Δy in the drawing. Nevertheless, this event indicated that her understanding of $\Delta y = m\Delta x$ was not entirely solid in certain situations.

In another task, the second task of Figure 3b, where she provided an estimate of $\sqrt[3]{1.05}$, she correctly found Δy by setting up $\frac{\Delta y}{.05} = \frac{1}{3}$ and expressed Δy as $\frac{1}{3} \cdot (.05)$. However, she mistakenly claimed that $\frac{1}{3} \cdot (.05)$ was the estimate of $\sqrt[3]{1.05}$, the same error she had made in previous tasks. It was only after being prompted to consider what she was looking for and where $\frac{1}{3} \cdot (.05)$ was in her drawing that she was able to coordinate the reference point (1, 1) with Δx and Δy to arrive at the correct approximation of $1 + \frac{1}{3} \cdot (.05)$.

Summary and Conclusion

In summary, prior to the intervention, Tanya exhibited several difficulties with slope and linear equations as described in other studies. Specifically, she held an action conception of slope, primarily as AR, and relied solely on the formula $\frac{y_2 - y_1}{x_2 - x_1}$ to determine slopes (Birgin, 2012; Cho & Nagle, 2017), even in situations where other approaches would be more effective. While she understood slope as GR, as $m = \frac{\Delta y}{\Delta x}$, she could only use it for integer Δx and Δy (Cho & Nagle, 2017; Thompson & Carlson, 2017). Additionally, she only had the slope-intercept form in her concept image (Vinner, 1983) of slope (Nagle & Moore-Russo, 2013) and was unable to construct an equation of a line in the slope-intercept form when the *y*-intercept was not explicitly provided (Cho & Nagle, 2017). Given her limited understanding of slope and line, Tanya would face challenges with calculus concepts such as derivatives, tangent line equations, and approximations of function values (David et al., 2019; Frank & Thompson, 2021; Nagle et al., 2022).

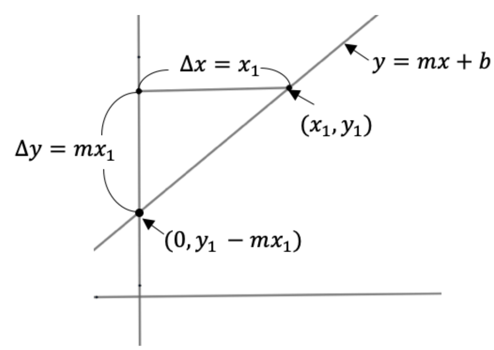
Tanya's understanding of slope and lines improved to some extent during the intervention. She extended her numerical understanding of ¼ = $\frac{\frac{1}{4}}{1}$ to a transition level of slope as GR and FP—conceiving $m = \frac{1}{4}$ as $\Delta y = 1$ for $\Delta x = 4$ as well as the unit rate of $\Delta y = \frac{1}{4}$ for $\Delta x = 1$. She also used the unit rate to understand Δy as $m\Delta x$ for integer Δx. For instance, she visualized Δy for $\Delta x = 3$ as an iteration of $\frac{1}{4}$ three times and wrote $\Delta y = 3\cdot\frac{1}{4}$ with no work. However, her interiorization of $\Delta y = m\Delta x$ was limited to integer Δx; for non-integer Δx, she obtained $\Delta y = m\Delta x$ as a result of computation from $m = \frac{\Delta y}{\Delta x}$ rather than as a multiple of m and Δx. Nagle et al. (2019) state that a process stage of slope involves an understanding of $\Delta y = m\Delta x$ as well as the connection between the geometric and algebraic representations of $\Delta y = m\Delta x$ and $y - y_1 = m(x - x_1)$ without explicit work. In Tanya's case, with her improved knowledge of $\Delta y = m\Delta x$, yet lower than the process stage, she successfully determined Δy and "located" points needed for linear equations or approximation in almost all tasks. The primary difficulty she had with linear equation and approximation tasks was determining the coordinates of the points.

When Tanya needed to consider a new point—to find the *y*-intercept or a point on a tangent line for approximation— she correctly made moves of Δx and Δy from the location of the reference point to the location of the new point. However, when she determined the coordinates of the new point, she often falsely claimed that Δy was the *y*-coordinate of the new point. According to Thompson et al. (2017), conceiving a point on a graph as a multiplicative object requires an understanding that coordinates of the point are a representation of "a state of two quantities' covariation" (p. 1307). In Tanya's case, she knew the location of the new point based on the location of the reference point by correctly applying the quantity changes of Δx and Δy encoded in the slope. However, she failed to incorporate Δx and Δy into the coordinates of the reference point to determine the coordinates of the new point, failing to conceive point as a multiplicative object.

Returning to the process stage of slope (Nagle et al., 2019), Tanya never made a direct connection between $\Delta y = m\Delta x$ and $y - y_1 = m(x - x_1)$ during the intervention. In fact, she never understood the point-slope form, $y - y_1 = m(x - x_1)$, or $m = \frac{y-y_1}{x-x_1}$, to the level that she could successfully use either form to construct linear equations by herself. Nevertheless, with her lower than process conception of slope, she could construct a linear equation in the slope-intercept form on some cases. For instance, when constructing an equation of a line passing through the point (x_1, y_1) with the slope of *m*, she set up the equation, $y = mx + b$, or $y = mx +$, with the blank replacing *b*, and made the horizontal and vertical moves of $\Delta x = x_1$ and $\Delta y = mx_1$ to locate the *y*-intercept (see Figure 10). Subsequently, she subtracted Δy from y_1 to determine the value of *b* as $y_1 - mx_1$ and constructed the linear equation, $y = mx + y_1 - mx_1$, a treated form of the point-slope form, $y - y_1 = m(x - x_1)$. Considering Tanya's case, one may make sense of the slope-intercept form, $y = mx + b$, with the *y*-intercept *b* as $y_1 - mx_1$, by combining the conceptions of slope and point as a multiplicative object.

Figure 10

Equation of Line that Passes through the Point (x_1, y_1) *with the Slope of m*



The findings of this study suggest that students' difficulties with tangent lines and approximations reported in previous research (Amit & Vinner, 1990; Biza & Zachariades, 2010; Çekmez & Baki, 2016; Orton, 1983; Moore-Russo & Nagle in this volume) may be linked to their insufficient understanding of critical ideas in linear equations, including slope and point. As shown in Tanya' case, students may not grasp the form, $y - f(x_1) = f'(x_1)(x - x_1)$, $y = f'(x_1)(x - x_1) + f(x_1)$, or $f'(x_1) = \frac{y-f(x_1)}{x-x_1}$, as a representation of a line passing through the point $(x_1, f(x_1))$ with a slope of $f'(x_1)$. Additionally, when utilizing a geometric approach to solve approximation problems, they may face challenges in determining $\Delta y = m\Delta x$, regardless of their ability to compute $m = f'(x)$, and in incorporating Δx and Δy into the coordinates of the reference point $(x_1, f(x_1))$ for the coordinates of new points. However, since the arguments are primarily based on a single student's experience, additional research is necessary to confirm them for a larger population.

I acknowledge that there are alternative approaches for constructing a linear equation requiring different kinds of understanding. For example, one can formulate an equation of a line passing through a point (x_1, y_1) with a slope of m by setting up the equation $y = mx + b$ and determining the value of b by substituting x_1 and y_1 into $y = mx + b$. Similarly, one can construct the tangent of the function $y = f(x)$ at $(a, f(a))$ by setting up the equation $y = f'(a)x + b$ and determining the value of b by substituting a and $f(a)$ into $y = f'(a)x + b$. The tangent line can then be used to approximate a function value $f(a + h)$ by substituting $a + h$ into x when h is close to 0. These techniques may require less cognitive demand (Stein & Smith, 1998) and therefore may be more accessible to some students. However, I believe that the geometric approach employed in my intervention has the potential to benefit students in some ways. It could help them connect algebraic and geometric representations of slope and point, conceive slope as a rate of change, understand coordinates as quantities of magnitudes, understand $\Delta y = m\Delta x$, and conceive point as a multiplicative object.

I conclude this study by describing a set of new and revised activities that I plan to implement with my college algebra and precalculus students, in addition to the activities presented in this study. As suggested by the findings of this study, it is crucial for students to engage in various opportunities to develop a deep understanding of point and slope. While these activities are designed for college students, I believe they may be also suitable for secondary students who possess an action conception of slope as AR and GR. The following section provides a detailed description of these activities.

Activities for Understanding Point, Slope, and Linear Equations

Activity 1

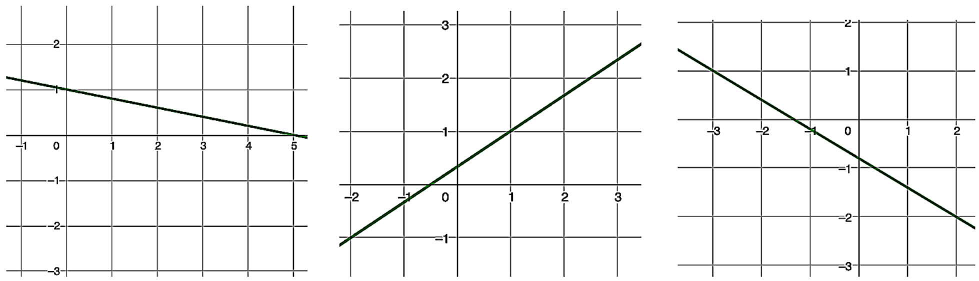
a) Plot some points (x, y) that satisfy the following condition: The slope between (1, 2) and (x, y) is 3. Construct the graph formed by all points that satisfy the condition and explain the graph, including why the graph has no holes.

b) Construct an equation representing the graph in part (a) and explain how the equation is related to the graph.
c) Redo parts (a) and (b) using the point $(-1, 3)$ instead of (1, 2) and a slope of -3 instead of 3.
d) Redo parts (a) and (b) using the point (x_1, y_1) instead of (1, 2) and a slope of m instead of 3.

Activity 1 is a conversion task from verbal descriptions to geometric and algebraic representations of lines. Students in this task construct a line graph consisting of points that maintain a constant slope of $\frac{\Delta y}{\Delta x}$ with a given point. They also construct a linear equation in the form of $m = \frac{y-y_1}{x-x_1}$. In part (a), students will develop a transition level of understanding of slope as GR, as they examine the directions and the magnitudes of various Δx and Δy. Initially, they may locate new points by drawing tick marks to trace the changes, Δx and Δy. However, as they work with different points, they are expected to eventually understand that $\Delta y = 3\Delta x$ for all Δx and conceptualize the new point as $(1+\Delta x, 2+3\Delta x)$ by incorporating Δx and $\Delta y = 3\Delta x$ into the reference point (1, 2). This way, students would conceive points as multiplicative objects, and furthermore, they would visualize slope using dynamic images of similar right triangles that result from varying *x* and *y*, especially while explaining the graph with no holes.

In part (b), students construct an equation $3 = \frac{y-2}{x-1}$ with a conception of slope as AR, initially treating (*x, y*) as a fixed point. However, as they explain $3 = \frac{y-2}{x-1}$ for all (*x, y*), they would reach a transition level of understanding of slope as AR and realize that $3 = \frac{y-2}{x-1}$ is true for all (*x, y*). Furthermore, students would connect AR and GR (a transition level of slope) and establish a connection between the line graph and equation. Moreover, the subtasks (c) and (d) would help students generalize their understanding of slope and equation by exploring a negative slope of -3 and an unknown slope of m. As a result, they would understand that a line graph is a collection of points forming a constant slope with a fixed point. Additionally, they would recognize that a linear equation can be represented in the form of $m = \frac{y-y_1}{x-x_1}$, which is an equivalent form of the line in point-slope form, $y - y_1 = m(x - x_1)$.

Activity 2



(a) Assuming the point (2, c) is on each of the three graphs above, determine the value of c in each graph.

(b) Use the strategy employed in Activity 1 to construct a line equation for each of the graphs above.

(c) A line equation can be represented in various forms, including $m = \frac{y-y_1}{x-x_1}$, $y - y_1 = m(x - x_1)$, and $y = mx + b$. Manipulate the equations constructed in part (b) to rewrite them in both $y - y_1 = m(x - x_1)$ and $y = mx + b$ forms.

(d) Explain what x, y, m, x_1, y_1, and b, in part (c) represent in graphs.

Activity 2 includes tasks that would help students conceive a point as a multiplicative object, strengthen their transition level of understanding of slope as FP, and establish connections between line graphs and various forms of linear equations. In part (a), students find the slopes of various line graphs, including lines with negative slopes, and use them to determine the coordinates of (2, c). While determining the value c, they would develop an understanding of the unit rate of change and conceive points as multiplicative objects. For example, in the first graph in part (a), they would understand $m = -\frac{1}{5}$ as $\Delta y = -\frac{1}{5}$ for $\Delta x = 1$ and find $\Delta y = 2 \cdot -\frac{1}{5}$ for $\Delta x = 2$. Subsequently, they determine the value of c by adding $2 \cdot -\frac{1}{5}$ to the y-coordinate of the reference point (0, 1), incorporating Δx and Δy into the coordinates of the reference point.

In parts (b) and (c), students may construct linear equations in the form of $m = \frac{y-y_1}{x-x_1}$ and transform them into both the $y - y_1 = m(x - x_1)$ form and the $y = mx + b$ form. However, since they may approach this process purely algebraically, they may not grasp the meanings of x, y, x_1, y_1, m, and b in relation to graphs; specifically, they may overlook the interpretation of b as the *y*-coordinate of the *y*-intercept in the relation $b = y_1 - mx_1$. The part (d)) aim to help students connect graphs and equations by translating the meanings of the sub-concepts of lines.

Activity 3

a) Plot points (*x, y*) that satisfy the following condition: The slope between (1, 2) and (*x, y*) is $3\Delta x$, where Δx represents the signed horizontal distance between (1, 2) and (*x, y*).
b) Construct the graph formed by all points that satisfy the condition in (a). Subsequently, construct an equation representing the graph.
c) Redo parts (a) and (b) using the point (-2, 3) instead of (1, 2) and the slope $-2\Delta x$ instead of $3\Delta x$.
d) Redo parts (a) and (b) using the point (*h, k*) instead of (1, 2) and the slope $a\Delta x$instead of $3\Delta x$.

Activity 3 is a construction activity focused on parabolic equations and graphs. While history and set theory define a parabola as a collection of points equidistant from a fixed point and a fixed line, I offer an alternative description of parabola emphasizing slope as a rate of change. This activity aims to help students understand the concept of slope beyond linear equations and construct a parabola equation in the vertex form, $y = a(x - h)^2 + k$.

In part (a), students may initially use the trial-and-error method to find some points that satisfy the given condition. However, as they progress, they may develop strategies to locate points that make the slope of $3\Delta x$. For instance, they may fix the *x*-variable of the point as a constant, such as (5, *y*), and calculate the slope $3\Delta x = 12$ with $\Delta x = 4$. They can then calculate $\Delta y = 48$ from the equation $\frac{\Delta y}{4} = 12$ using their understanding of slope as GR. Subsequently, they would incorporate 48 into the *y*-coordinate of (1, 2) to determine the coordinates of a new point $(5, 50)$. While repeating this process for various *x*-values, they would become aware that they can find a point (*x, y*) that satisfies the given condition for any *x*. By applying the same method, they can locate new points and determine the coordinates of these points as (1 + Δx, 2 + Δy) with $\Delta y = 3\Delta x^2$ for some Δx.

In part (b), students come to realize that for any point (*x, y*) that satisfies the given condition, $\Delta x = x - 1$ and $\Delta y = y - 2$ by connecting slope as GR and AR. Accordingly, they set up the equation $\frac{y-2}{x-1} = 3(x - 1)$ derived from $\frac{\Delta y}{\Delta x} = 3\Delta x$. They then transform the equation into the form $y - 2 = 3(x - 1)^2$, and further into $y = 3(x - 1)^2 + 2$, a parabola in the vertex form. In doing so, they establish a connection between the parabola equation and graph while understanding the characteristics of the parabola, such as its symmetry along the line $x = 1$ and its minimum value of 2 at $x = 1$. Subsequently, in parts (c) and (d), students generalize the situation and construct an equation of the form $y = a(x - h)^2 + k$. Moreover, they develop an understanding that the sign of a determines the shape of the graph, whether it is an upward parabola or a downward parabola. If $a > 0$, the points satisfying the given condition lie above the point (h, k), as $\Delta y = a\Delta x^2$ is positive regardless of the sign of Δx. Similarly, if $a < 0$, the points satisfying the given condition lie below the point (h, k) as $\Delta y = a\Delta x^2$ is negative regardless of the sign of Δx.

References

Abramson, J. (2018). *Algebra and trigonometry*. OpenStax.

Amit, M., & Vinner, S. (1990). Some misconceptions in calculus: Anecdotes or the tip of the iceberg? In G. Booker, P. Cobb & T. N. de Mendicuti (Eds.), *Proceedings of the 14th International Conference for the Psychology of Mathematics Education*, (Vol. 1, pp. 3-10). Oaxtepec, Mexico: Cinvest.

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. Springer Science & Business Media.

Barr, G. (1980). Graphs, gradients and intercepts. *Mathematics in School*, *9*(1), 5-6.

Birgin, O. (2012). Investigation of eighth-grade students' understanding of the slope of the linear function. *Bolema: Boletim de Educação Matemática*, *26*, 139-162.

Biza, I., & Zachariades, T. (2010). First year mathematics undergraduates' settled images of tangent line. *The Journal of Mathematical Behavior*, *29*(4), 218-229.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378.

Çekmez, E., & Baki, A. (2016). Examining students' generalizations of the tangent concept: A theoretical perspective. *PRIMUS*, *26*(5), 466-484.

Cho, P., & Nagle, C. (2017). Procedural and conceptual difficulties with slope: An analysis of students' mistakes on routine tasks. *International Journal of Research in Education and Science*, *3*(1), 135-150.

David, E. J., Roh, K. H., & Sellers, M. E. (2019). Value-thinking and location-thinking: Two ways students visualize points and think about graphs. *The Journal of Mathematical Behavior*, *54*, 100675.

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior*, *39*, 135-155.

Frank, K. (2016). Plotting points: Implications of “over and up” on students’ covariational reasoning. In Wood, M.B., Turner, E. E., Civil, M., & Eli, J.A. (Eds.), *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, pp. 573-580. Tucson, AZ: The University of Arizona.

Frank, K., & Thompson, P. W. (2021). School students’ preparation for calculus in the United States. *ZDM–Mathematics Education*, *53*, 549-562.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge: University Press.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.

Herscovics, N. & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, *27*, 59-78.

Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior*, *31*(3), 313-330.

Kerslake, D. (1981). Graphs. *Children’s understanding of mathematics*, *11*(16), 120-136.

Knuth, E. (2000). Student understanding of the Cartesian coordinate connection: An exploratory study. *Journal for Research in Mathematics Education*, *31*(4), 500-508.

Kondratieva, M. F.; Radu, O. (2009) Fostering connections between the verbal, algebraic, and geometric representations of basic planar curves for student’s success in the study of mathematics. *The Mathematics Enthusiast*, *6*(1), 213-238.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from* (Vol. 6). New York: Basic Books.

Lobato, J., Ellis, A. B., & Muñoz, R. (2003). How" focusing phenomena" in the instructional environment support individual students' generalizations. *Mathematical Thinking and Learning*, *5*(1), 1-36.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook.* Thousand Oaks, CA: SAGE Publications, Inc.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 782-789). Pittsburgh: RUME.

Moore-Russo, D., Conner, A., & Rugg, K. I. (2011). Can slope be negative in 3-space? Studying concept image of slope through collective definition construction. *Educational Studies in Mathematics*, *76*(1), 3-21.

Moschkovich, J. N. (1996). Moving up and getting steeper: Negotiating shared descriptions of linear graphs. *The Journal of the Learning Sciences*, *5*(3), 239-277.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations, and connections among them. In T. Romberg, E. Fennema, & T. Carpenter (Eds.), *Integrating research on the graphical representation of function* (pp. 69-100). Hillsdale, NJ: Erlbaum.

Nagle, C., & Moore-Russo, D. (2013). The concept of slope: Comparing teachers' concept images and instructional content. *Investigations in Mathematics Learning*, *6*(2), 1-18.

Nagle, C., Moore-Russo, D., & Fisher, T. (2022). A Study of slope explanations across a standards-based textbook series. *International Journal for Mathematics Teaching and Learning*, *23*(1), 25-45.

Moore-Russo, D. & Nagle, C. (in this volume). Connecting representations and ways of thinking about slope from algebra to calculus.

Nagle, C., Martínez-Planell, R., & Moore-Russo, D. (2019). Using APOS theory as a framework for considering slope understanding. *The Journal of Mathematical Behavior*, *54*,100684.

Orton, A. (1983). Student's understanding of differentiation. *Educational Studies in Mathematics*, *14*(3), 235-250.

Padilla, M., Mckenzie, D. L., & Shaw, E. L. (1986). An examination of the line graphing ability of students in grades seven through twelve. *School Science and Mathematics*, *86*(1), 20-26.

Saldanha, L., & Thompson, P. W. (1998). Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah, K. R. Dawkins, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the*

Psychology of Mathematics Education (Vol. 1, pp. 298-304). Raleigh, NC: North Carolina State University.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Sierpinska, A. (1985). Obstacles épistémologiques relatifs à la notion de limite. *Recherches en Didactique des Mathématiques (Revue)*, *6*(1), 5-67.

Smith, E., & Confrey, J. (1994). Multiplicative structures and the development of logarithms: What was lost by the invention of function. *The development of multiplicative reasoning in the learning of mathematics*, 331-360.

Stalvey, H. E., & Vidakovic, D. (2015). Students' reasoning about relationships between variables in a real-world problem. *The Journal of Mathematical Behavior, 40*, 192–210.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*, 267-306.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School*, *3*(4), 268-275.

Stevens, I. E., Paoletti, T., Moore, K. C., Liang, B., & Hardison, H. (2017). Principles for designing tasks that promote covariational reasoning. In *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 928-936).

Stump, S. L. (2001). High school precalculus students' understanding of slope as measure. *School Science and Mathematics*, *101*(2), 81-89.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *Journal of Mathematical Behavior*, *48*, 95-111.

Trigueros, M., & Martinez-Planell, R. (2010). Geometrical representations in the learning of two variable functions. *Educational Studies in Mathematics*, *73*, 3-19.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, *14*(3), 293-305.