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ANALYTICAL DETERMINATION OF COUPLED BENDING-TORSION VIBRATIONS OF CANTILEVER BEAMS BY MEANS OF STATION FUNCTIONS

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SUMMARY

A method based on the concept of Station Functions is presented for calculating the modes and the frequencies of nonuniform cantilever beams vibrating in torsion, bending, and coupled bending-torsion motion. The method combines some of the advantages of the Rayleigh-Ritz and Stodola methods, in that a continuous loading function for the beam is used, with the advantages of the influence-coefficient method, in that the continuous loading function is obtained in terms of the displacements at a finite number of stations along the beam.

The Station Functions were derived for a number of stations ranging from one to eight. The deflections were obtained in terms of the physical properties of the beam and Station Numbers, which are general in nature and which have been tabulated for easy reference. Examples were worked out in detail; comparisons were made with exact theoretical results. For a uniform cantilever beam, for n stations along the beam, the first $n-1$ modes and frequencies were in excellent agreement with the theoretically exact values. It was shown that the effect of coupling between bending and torsion is to reduce the first natural frequency to a value below that which it would have if there were no coupling.

INTRODUCTION

The failure of turbine and compressor blades due to vibrations has led to an increased interest in the study of the vibrations of these blades and in the determination of the natural modes and frequencies. In such theoretical studies, it is usually assumed that the compressor or turbine blade acts as a cantilever beam. The calculation of the uncoupled modes of arbitrarily shaped cantilever beams has been extensively investigated (references 1 to 4), but little work has as yet been done on calculating the coupled modes of such beams. If the geometry of the beam is such that coupling exists, the coupled modes are the actual vibrational modes that must be calculated.

Four general methods are currently in use for calculating uncoupled modes and frequencies of nonuniform beams. These methods are the Rayleigh-Ritz or energy method (reference 1), the Stodola method (references 5 and 6), the influence-coefficient method (references 4 and 7), and the integral-equation method (references 8 and 9). For each of these methods, computational work can usually be carried out in several ways. For example, by the use of influence coefficients the modes and frequencies can be determined by Mykelstad's iteration procedure (reference 7) or by matrix methods (reference 4).

Any one of these methods can be extended to the calculation of coupled bending-torsion modes. The Rayleigh-Ritz method usually requires that the uncoupled modes be determined before the coupled modes can be computed. In applying either the Rayleigh-Ritz or the Stodola method, great difficulty is encountered in accurately determining the higher modes, because the lower modes must first be "swept out" by the use of exact orthogonality conditions (reference 10); the process will otherwise always converge back to the lowest mode. The same difficulties are encountered in the integral-equation method.

The influence-coefficient method avoids these difficulties by reducing the problem to one having a finite number of degrees of freedom. The beam is divided into n intervals and a concentrated loading is assumed at the center of gravity of each interval. The solution of the resultant determinantal equation gives the first n modes. The accuracy of the higher modes is, however, very poor. Only the first third of the modes and the first half of the frequencies are obtained within the usual engineering accuracy. Carrying along so many useless modes greatly increases the labor involved.

A straightforward, accurate method for determining the coupled bending-torsion modes and the frequencies of nonuniform cantilever beams, together with applications of this method, was developed at the NACA Lewis laboratory and is presented herein. This method is based on the use of Station Functions as first discussed in reference 11. Incorporated in the method are the advantages of the continuous-function deflections of the Rayleigh-Ritz and Stodola methods together with the advantages of the finite number of degrees of freedom of the influence-coefficient method. When the method is applied to a uniform beam, the first n-1 roots of the resultant determinantal equation are amply accurate for engineering purposes.

The final determinantal equation is solved herein by matrix-iteration methods. Any other convenient method may, however, be used and no knowledge of matrix algebra is needed to carry out the calculations by the matrix method. The work can be done by an inexperienced computer, as the only operations necessary for determining each mode are cumulative multiplication and division. In addition, for the case in which the coupling coefficient remains constant along the beam, a simple quadratic formula and a series of curves are presented for determining the first coupled mode in terms of the uncoupled modes. Examples are developed in detail and comparisons with exact theoretical results are included.

THEORY

In the usual influence-coefficient methods for solving dynamical problems, a continuous body having an infinite number of degrees of freedom is replaced by a body having a finite number of degrees of freedom. Two principal assumptions are then made, which introduce inaccuracies into the solutions, particularly in the higher modes: (1) The resultant of the inertia loads of all the infinitesimal masses in a finite interval passes through the center of gravity of that interval; and (2) a concentrated load that is the resultant of a distributed load produces the same deflection as the distributed load. An attempt has been made to reduce the error due to the second of these assumptions by the use of weighting matrices (reference 12). Although the accuracy is thereby increased, the effect of the first assumption is still great enough to introduce serious errors (reference 11).

In order to eliminate these assumptions, Rauscher (reference 11) introduced the concept of Station Functions. Instead of assuming the inertia loads to be concentrated at the centers of gravity of the intervals, the inertia loads and, consequently, the deflections are assumed to be continuous functions along the beam. The values of these continuous deflection functions at the reference stations must equal the deflections of the reference stations. The loading on the beam is therefore a continuous function of the deflections of the reference stations. Inasmuch as the deflections of the reference stations can be computed from the loading on the beam, which in turn is available from the deflections, the deflections are therefore obtained as functions of themselves. This procedure gives n homogenous equations in the n deflections of the reference stations. The resultant determinantal equation has n roots for the frequency; it will be shown that at least n-1 of these roots are sufficiently accurate for engineering purposes if the deflection functions are properly chosen. (For coupled bending-torsion vibrations, 2n homogeneous equations and 2n roots are obtained for n stations.)

The deflection functions used must satisfy the boundary conditions of the problem and also the condition that, at any reference station, the value of the function must equal the deflection of the reference station. Although it is always possible to find directly a single function that will satisfy these conditions, it is more convenient to obtain different component functions at each station and to add all these component functions together to give the complete deflection function. Rauscher (reference 11) calls these component deflection functions Station Functions. For example, the complete torsional deflection function for the beam will have the following form:

$$\theta(z) = \sum_{j=1}^{n} f_j(z)\ \theta_j$$

where

z dimensionless distance along beam

$\theta(z)$ torsional deflection at distance z from root

θ_j torsional deflection at the j^{th} station

$f_j(z)$ Station Function in torsion associated with the j^{th} station (All symbols are defined in appendix A.)

Each Station Function must satisfy the boundary conditions of the problem and the following additional conditions: (1) At the reference station with which it is associated, the Station Function equals the deflection of that reference station; and (2) at all other reference stations, the Station Function equals zero. The sum of all these Station Functions will then give the complete deflection function for the beam. The Station Functions and corresponding loading functions are derived in appendix B for torsional vibrations, bending vibrations, and coupled bending-torsion vibrations of an arbitrary cantilever beam.

Torsional vibrations. - It is shown in appendix B that the torsional deflections of the reference stations for a beam divided into n intervals of length δ, as shown in figure 1, are given by the following system of equations:

$$\theta_i = \omega^2 \delta^2 \frac{I_0}{C_0} \sum_{j=1}^{n} \alpha_{ij} \, \theta_j \tag{1}$$

where

$$\alpha_{ij} \equiv \sum_{k=1}^{i} \frac{1}{C_k} \left[I_k N_{jk} - (k-1) I_k M_{jk} + \sum_{r=k+1}^{n} I_r M_{jr} \right] \tag{2}$$

$$i \text{ and } j = 1,2, \ . \ . \ . \ n$$

ω frequency of vibration

δ length of interval

I_0 mass moment of inertia per unit length about elastic axis at root section

I_k ratio of average mass moment of inertia per unit length of k^{th} interval to mass moment of inertia per unit length at root section

C_0 torsional stiffness of the root section

C_k ratio of average torsional stiffness of k^{th} interval to torsional stiffness at root section

The Station Numbers N_{jk} and M_{jk} are functions only of the integers k, j, and n and are defined as

$$\left.\begin{aligned} N_{jk} &\equiv \int_{k-1}^{k} z f_j(z) dz \\ M_{jk} &\equiv \int_{k-1}^{k} f_j(z) dz \end{aligned}\right\} \tag{3}$$

where $f_j(z)$ represents the Station Functions derived in appendix B and is given by

$$f_j(z) = a_{1j}\, z + a_{2j}\, z^2 + \dots + a_{(n+1)j}\, z^{(n+1)} \qquad (4)$$

The coefficients a_{ij} are determined in appendix B by satisfying the conditions on the Station Functions. The integrals in equations (3) are thus seen to be integrals of simple polynomials and the limits of integration are integers. The Station Numbers N_{jk} and M_{jk} are therefore rational numbers, functions only of the integers n, k, and j. These numbers have been evaluated and and are listed in tables I to VIII.

If the physical properties of the beam under consideration are known for each of the n intervals, C_k and I_k will be known. The Station Numbers N_{jk} and M_{jk} can be obtained from tables I to VIII. From equation (2), α_{ij} can then be easily calculated.

Equation (1) actually represents n homogeneous equations in the n unknown deflections θ_i. With $\frac{1}{\omega^2}\frac{C_0}{I_0\delta^2} \equiv \lambda$, these equations can be written as follows:

$$\left.\begin{aligned}
(\alpha_{11}-\lambda)\,\theta_1 + \alpha_{12}\theta_2 + \alpha_{13}\theta_3 + \dots + \alpha_{1n}\theta_n &= 0 \\
\alpha_{21}\,\theta_1 + (\alpha_{22}-\lambda)\,\theta_2 + \alpha_{23}\theta_3 + \dots + \alpha_{2n}\theta_n &= 0 \\
\alpha_{31}\,\theta_1 + \alpha_{32}\,\theta_2 + (\alpha_{33}-\lambda)\,\theta_3 + \dots + \alpha_{3n}\theta_n &= 0 \\
\dots\dots\dots\dots\dots\dots\dots\dots & \\
\alpha_{n1}\theta_1 + \alpha_{n2}\,\theta_2 + \alpha_{n3}\,\theta_3 + \dots + (\alpha_{nn}-\lambda)\,\theta_n &= 0
\end{aligned}\right\} \qquad (5)$$

For a nontrivial solution, the determinant of the coefficients must vanish and the characteristic equation becomes

$$\begin{vmatrix}
\alpha_{11}-\lambda & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22}-\lambda & \alpha_{23} & \dots & \alpha_{2n} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}-\lambda & \dots & \alpha_{3n} \\
\dots & \dots & \dots & \dots & \dots \\
\alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \dots & \alpha_{nn}-\lambda
\end{vmatrix} = 0 \qquad (6)$$

or

$$\left| \lambda I - [\alpha_{ij}] \right| = 0 \tag{6a}$$

where I is the identity matrix, and $[\alpha_{ij}]$ is the dynamical matrix.

Equation (6) can be solved for the n values of λ by any method available. The method used herein was to obtain the values of λ as the latent roots of the matrix $[\alpha_{ij}]$, which is actually the dynamical matrix for the problem. The mode shapes are obtained at the same time.

Bending vibrations. - The bending deflections for the beam shown in figure 1 are given by the following system of equations (see appendix B):

$$y_i = \omega^2 \delta^4 \frac{m_0}{B_0} \sum_{j=1}^{n} \beta_{ij}\, y_j \tag{7}$$

where

$$\beta_{ij} \equiv \sum_{k=1}^{i} \frac{1}{B_k} \left(m_k (iP'_{jk} - Q'_{jk}) + \sum_{r=k+1}^{n} m_r \left\{ (i-k+\frac{1}{2}) N'_{jr} + \left[\frac{k^3-(k-1)^3}{3} - \frac{(2k-1)i}{2} \right] M'_{jr} \right\} \right) \tag{8}$$

$$i \text{ and } j = 1, 2, \ldots n$$

m_0 mass per unit length of beam at root section

m_k ratio of average mass per unit length of k^{th} interval to mass per unit length at root section

B_0 bending stiffness at root section

B_k ratio of average bending stiffness of k^{th} interval to bending stiffness at root section

The Station Numbers M'_{jk}, N'_{jk}, P'_{jk}, and Q'_{jk} are functions only of the integers k, j, and n and are defined by

$$\left.\begin{aligned}
P'_{jk} &\equiv \int_{k-1}^{k} \left[\frac{z^2}{2} - (k-1)\,z + \frac{1}{2}(k-1)^2\right] g_j(z)dz \\
Q'_{jk} &\equiv \int_{k-1}^{k} \left[\frac{z^3}{6} - \frac{1}{2}(k-1)^2\,z + \frac{1}{3}(k-1)^3\right] g_j(z)dz \\
M'_{jk} &\equiv \int_{k-1}^{k} g_j(z)dz \\
N'_{jk} &\equiv \int_{k-1}^{k} z\,g_j(z)dz
\end{aligned}\right\} \quad (9)$$

The Station Functions $g_j(z)$ are derived in appendix B and are given by

$$g_j(z) = b_{2j}\,z^2 + b_{3j}\,z^3 + b_{4j}\,z^4 + \ldots + b_{(n+3)j}\,z^{(n+3)} \quad (10)$$

The integrals in equations (9) are thus seen to be integrals of simple polynomials. The Station Numbers M'_{jk}, N'_{jk}, P'_{jk}, and Q'_{jk} are rational numbers, functions only of the integers j, k, and n. These numbers have been evaluated and are listed in tables I to VIII.

If the physical properties of the beam are known for each of the n intervals, m_k and B_k will be known. The Station Numbers M'_{jk}, N'_{jk}, P'_{jk}, and Q'_{jk} are obtained from tables I to VIII; β_{ij} can then easily be calculated by equation (8).

The determinantal equation is